THE GEOMETRIC MEAN IS A BERNSTEIN FUNCTION

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ABSTRACT. In the paper, the authors establish, by using Cauchy integral formula in the theory of complex functions, an integral representation for the geometric mean of n positive numbers. From this integral representation, the geometric mean is proved to be a Bernstein function and a new proof of the well known AG inequality is provided.

1. Introduction

We recall some notions and definitions.

Definition 1.1 ([15, 26])**.** A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(t) \ge 0 \tag{1.1}$$

for $x \in I$ and $n \ge 0$.

The class of completely monotonic functions on $(0, \infty)$ is characterized by the famous Hausdorff-Bernstein-Widder Theorem below.

Proposition 1.1 ([26, p. 161, Theorem 12b])**.** *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that*

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\alpha(t), \tag{1.2}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

Definition 1.2 ([18, 20])**.** A function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(t)]^{(k)} \ge 0 \tag{1.3}$$

for $k \in \mathbb{N}$ on I.

It has been proved in [3, 8, 18, 20] that a logarithmically completely monotonic function on an interval I must be completely monotonic on I.

Definition 1.3 ([24, 26])**.** A function $f : I \subseteq (-\infty, \infty) \to [0, \infty)$ is called a Bernstein function on I if $f(t)$ has derivatives of all orders and $f'(t)$ is completely monotonic on I.

The class of Bernstein functions can be characterized by

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Proposition 1.2 ([24, p. 15, Theorem 3.2]). *A function $f : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it admits the representation*

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) \mathrm{d}\mu(t), \tag{1.4}$$

where $a, b \ge 0$ and μ is a measure on $(0,\infty)$ satisfying

$$\int_0^\infty \min\{1,t\} \,\mathrm{d}\mu(t) < \infty.$$

In [5, pp. 161–162, Theorem 3] and [24, p. 45, Proposition 5.17], it was discovered that the reciprocal of any Bernstein function is logarithmically completely monotonic.

Definition 1.4 ([1]). If $f^{(k)}(t)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(t)$ is not completely monotonic on I, then $f(t)$ is called a completely monotonic function of k-th order on an interval I.

It is obvious that a completely monotonic function of first order is a Bernstein function if and only if it is nonnegative on I.

Definition 1.5 ([24, p. 19, Definition 2.1]). If $f : (0,\infty) \to [0,\infty)$ can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} \mathrm{d}\mu(s), \tag{1.5}$$

then it is called a Stieltjes function, where $a, b \ge 0$ are nonnegative constants and μ is a nonnegative measure on $(0,\infty)$ such that

$$\int_0^\infty \frac{1}{1+s} \,\mathrm{d}\mu(s) < \infty.$$

The set of logarithmically completely monotonic functions on $(0,\infty)$ contains all Stieltjes functions, see [3] or [22, Remark 4.8]. In other words, all the Stieltjes functions are logarithmically completely monotonic on $(0,\infty)$.

In the newly-published paper [7], a new notion "completely monotonic degree" of nonnegative functions was naturally introduced and initially studied.

We also recall that the extended mean value $E(r,s;x,y)$ may be defined as

$$E(r,s;x,y) = \left[\frac{r(y^s - x^s)}{s(y^r - x^r)}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y) \ne 0; \tag{1.6}$$

$$E(r,0;x,y) = \left[\frac{y^r - x^r}{r(\ln y - \ln x)}\right]^{1/r}, \qquad r(x-y) \ne 0; \tag{1.7}$$

$$E(r,r;x,y) = \frac{1}{e^{1/r}} \left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, \qquad r(x-y) \ne 0; \tag{1.8}$$

$$E(0,0;x,y) = \sqrt{xy}\,, \qquad x \ne y; \tag{1.9}$$

$$E(r,s;x,x) = x, \qquad x = y;$$

where x and y are positive numbers and $r, s \in \mathbb{R}$. Because this mean was first defined in [25], so it is also called Stolarsky's mean. Many special mean values with two variables are special cases of E, for example,

$$E(r,2r;x,y) = M_r(x,y), \qquad \text{(power mean or Hölder mean)}$$

$$E(1,p;x,y) = L_p(x,y), \qquad \text{(generalized or extended logarithmic mean)}$$

$$\begin{aligned}
E(1,1;x,y)&=I(x,y), && \text{(identric or exponential mean)}\\
E(1,2;x,y)&=A(x,y), && \text{(arithmtic mean)}\\
E(0,0;x,y)&=G(x,y), && \text{(geometric mean)}\\
E(-2,-1;x,y)&=H(x,y), && \text{(harmonic mean)}\\
E(0,1;x,y)&=L(x,y). && \text{(logarithmic mean)}
\end{aligned}$$

For more information on E, please refer to the monograph [4], the papers [9, 10, 11], and closely-related references therein.

It is easy to see that the arithmetic mean

$$A_{x,y}(t)=A(x+t,y+t)=A(x,y)+t$$

is a trivial Bernstein function of $t\in(-\min\{x,y\},\infty)$ for $x,y>0$.

It is not difficult to see that the harmonic mean

$$H_{x,y}(t)=H(x+t,y+t)=\frac{2}{\frac{1}{x+t}+\frac{1}{y+t}} \tag{1.10}$$

for $t\in(-\min\{x,y\},\infty)$ and $x,y>0$ with $x\ne y$ meets

$$H'_{x,y}(t)=\frac{2\bigl[x^2+y^2+2(x+y)t+2t^2\bigr]}{(x+y+2t)^2}=1+\frac{(x-y)^2}{(x+y+2t)^2}>1. \tag{1.11}$$

It is obvious that the derivative $H'_{x,y}(t)$ is completely monotonic with respect to t. As a result, the harmonic mean $H_{x,y}(t)$ is a Bernstein function of t on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$.

In [21, Remark 6], it was pointed out that the reciprocal of the identric mean

$$I_{x,y}(t)=I(x+t,y+t)=\frac{1}{e}\biggl[\frac{(x+t)^{x+t}}{(y+t)^{y+t}}\biggr]^{1/(x-y)} \tag{1.12}$$

for $x,y>0$ with $x\ne y$ is a logarithmically completely monotonic function of $t\in(-\min\{x,y\},\infty)$ and that the identric mean $I_{x,y}(t)$ for $t>-\min\{x,y\}$ with $x\ne y$ is also a completely monotonic function of first order (that is, a Bernstein function).

In [17, p. 616], it was concluded that the logarithmic mean

$$L_{x,y}(t)=L(x+t,y+t) \tag{1.13}$$

is increasing and concave in $t>-\min\{x,y\}$ for $x,y>0$ with $x\ne y$. More strongly, it was proved in [19, Theorem 1] that the logarithmic mean $L_{x,y}(t)$ for $x,y>0$ with $x\ne y$ is a completely monotonic function of first order in $t\in(-\min\{x,y\},\infty)$, that is, the logarithmic mean $L_{x,y}(t)$ is a Bernstein function of $t\in(-\min\{x,y\},\infty)$.

Recently, the geometric mean

$$G_{x,y}(t)=G(x+t,y+t)=\sqrt{(x+t)(y+t)} \tag{1.14}$$

was proved in [23] to be a Bernstein function of t on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$, and its integral representation

$$G_{x,y}(t)=G(x,y)+t+\frac{x-y}{2\pi}\int_0^\infty\frac{\rho((x-y)s)}{s}e^{-ys}\bigl(1-e^{-st}\bigr)\,\mathrm{d}s \tag{1.15}$$

for $x > y > 0$ and $t > -y$ was discovered, where

$$\begin{aligned}\rho(s) &= \int_0^{1/2} q(u)\bigl[1 - e^{-(1-2u)s}\bigr]e^{-us}\,\mathrm{d}u \\ &= \int_0^{1/2} q\biggl(\frac{1}{2} - u\biggr)\bigl(e^{us} - e^{-us}\bigr)e^{-s/2}\,\mathrm{d}u \\ &\ge 0\end{aligned} \tag{1.16}$$

on $(0,\infty)$ and

$$q(u) = \sqrt{\frac{1}{u} - 1} - \frac{1}{\sqrt{1/u - 1}} \tag{1.17}$$

on $(0,1)$.

Let $a = (a_1, a_2, \dots, a_n)$ for $n \in \mathbb{N}$, the set of all positive integers, be a given sequence of positive numbers. Then the arithmetic and geometric means $A_n(a)$ and $G_n(a)$ of the numbers $a_1, a_2, \dots, a_n$ are defined respectively as

$$A_n(a) = \frac{1}{n}\sum_{k=1}^{n} a_k \tag{1.18}$$

and

$$G_n(a) = \left(\prod_{k=1}^{n} a_k\right)^{1/n}. \tag{1.19}$$

It is general knowledge that

$$G_n(a) \le A_n(a), \tag{1.20}$$

with equality if and only if $a_1 = a_2 = \dots = a_n$.

There has been a large number, presumably over one hundred, of proofs of the AG inequality (1.20) in the mathematical literature. The most complete information, so far, can be found in the monographs [2, 4, 12, 13, 14, 16] and a lot of references therein.

In this paper, we establish, by using Cauchy integral formula in the theory of complex functions, an integral representation of the geometric mean

$$G_n(a+z) = \left[\prod_{k=1}^{n}(a_k + z)\right]^{1/n}, \tag{1.21}$$

where $a = (a_1, a_2, \dots, a_n)$ satisfies $a_k > 0$ for $1 \le k \le n$ and

$$z \in \mathbb{C} \setminus (-\infty, -\min\{a_k, 1 \le k \le n\}].$$

From this integral representation, it is immediately derived that the geometric mean $G_n(a+t)$ for $t \in (-\min\{a_k, 1 \le k \le n\}, \infty)$ is a Bernstein function, where $a + t = (a_1 + t, a_2 + t, \dots, a_n + t)$, and a new proof of the AG inequality (1.20) is provided.

2. Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (Cauchy integral formula [6, p. 113])**.** *Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then*

$$f(z)=\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}\,\mathrm{d}w,\quad z\in D. \tag{2.1}$$

Lemma 2.2. *For $z\in\mathbb{C}\setminus(-\infty,-\min\{a_k,1\le k\le n\}]$ with $a=(a_1,a_2,\dots,a_n)$ and $a_k>0$, the principal branch of the complex function*

$$f_{a,n}(z)=G_n(a+z)-z, \tag{2.2}$$

where $a+z=(a_1+z,a_2+z,\dots,a_n+z)$, meets

$$\lim_{z\to\infty}f_{a,n}(z)=A_n(a). \tag{2.3}$$

Proof. By L'Hôspital's rule in the theory of complex functions, we have

$$\begin{aligned}\lim_{z\to\infty}f_{a,n}(z)&=\lim_{z\to\infty}\left\{z\left[G_n\left(1+\frac{a}{z}\right)-1\right]\right\}\\&=\lim_{z\to0}\frac{G_n(1+az)-1}{z}=\lim_{z\to0}\frac{\mathrm{d}}{\mathrm{d}z}\left[\prod_{k=1}^{n}(1+a_kz)\right]^{1/n}=A_n(a),\end{aligned}$$

where $1+\frac{a}{z}=\left(1+\frac{a_1}{z},1+\frac{a_2}{z},\dots,1+\frac{a_n}{z}\right)$ and $1+az=(1+a_1z,1+a_2z,\dots,1+a_nz)$. Lemma 2.2 is thus proved. □

Lemma 2.3. *Let $a=(a_1,a_2,\dots,a_n)$ with $a_k>0$ for $1\le k\le n$ and let $[a]$ be the rearrangement of the positive sequence a in an ascending order, that is, $[a]=\left(a_{[1]},a_{[2]},\dots,a_{[n]}\right)$ and $a_{[1]}\le a_{[2]}\le\dots\le a_{[n]}$. For $z\in\mathbb{C}\setminus(-\infty,0]$, let*

$$h_n(z)=G_n\left([a]-a_{[1]}+z\right)-z, \tag{2.4}$$

where $[a]-a_{[1]}+z=\left(z,a_{[2]}-a_{[1]}+z,\dots,a_{[n]}-a_{[1]}+z\right)$. Then the principal branch of $h_n(z)$ satisfies

$$\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)=\begin{cases}\left[\prod_{k=1}^{n}\left|a_{[k]}-a_{[1]}-t\right|\right]^{1/n}\sin\frac{\ell\pi}{n}, & t\in\left(a_{[\ell]}-a_{[1]},a_{[\ell+1]}-a_{[1]}\right]\\0, & t\ge a_{[n]}-a_{[1]}\end{cases} \tag{2.5}$$

for $1\le\ell\le n-1$.

Proof. For $t\in(0,\infty)\setminus\left\{a_{[\ell+1]}-a_{[1]},1\le\ell\le n-1\right\}$ and $\varepsilon>0$, we have

$$\begin{aligned}h_n(-t+i\varepsilon)&=G_n\left([a]-a_{[1]}-t+i\varepsilon\right)+t-i\varepsilon\\&=\exp\left[\frac1n\sum_{k=1}^{n}\ln\left(a_{[k]}-a_{[1]}-t+i\varepsilon\right)\right]+t-i\varepsilon\\&=\exp\left\{\frac1n\sum_{k=1}^{n}\left[\ln\left|a_k-a_{[1]}-t+i\varepsilon\right|+i\arg\left(a_{[k]}-a_{[1]}-t+i\varepsilon\right)\right]\right\}+t-i\varepsilon\end{aligned}$$

$$
\to \begin{cases} \exp\left(\dfrac{1}{n}\sum_{k=1}^{n}\ln\left|a_{[k]}-a_{[1]}-t\right|+\dfrac{\ell\pi}{n}i\right)+t, t\in\left(a_{[\ell]}-a_{[1]},a_{[\ell+1]}-a_{[1]}\right)\\ \exp\left(\dfrac{1}{n}\sum_{k=1}^{n}\ln\left|a_{[k]}-a_{[1]}-t\right|+\pi i\right)+t, t>a_{[n]}-a_{[1]} \end{cases}
$$

$$
= \begin{cases} \left(\displaystyle\prod_{k=1}^{n}\left|a_{[k]}-a_{[1]}-t\right|\right)^{1/n}\exp\left(\dfrac{\ell\pi}{n}i\right)+t, t\in\left(a_{[\ell]}-a_{[1]},a_{[\ell+1]}-a_{[1]}\right)\\ \left(\displaystyle\prod_{k=1}^{n}\left|a_{[k]}-a_{[1]}-t\right|\right)^{1/n}\exp(\pi i)+t, t>a_{[n]}-a_{[1]} \end{cases}
$$

as $\varepsilon\to0^+$. As a result, we have

$$
\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)= \begin{cases} \left(\displaystyle\prod_{k=1}^{n}\left|a_{[k]}-a_{[1]}-t\right|\right)^{1/n}\sin\dfrac{\ell\pi}{n}, & t\in\left(a_{[\ell]}-a_{[1]},a_{[\ell+1]}-a_{[1]}\right);\\ 0, & t>a_{[n]}-a_{[1]}. \end{cases}
$$

For $t=a_{[\ell+1]}-a_{[1]}$ for $1\le\ell\le n-1$, we have

$$
\begin{aligned}
h_n(-t+i\varepsilon)&=\exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\left(a_{[k]}-a_{[1]}-t+i\varepsilon\right)+\frac{1}{n}\ln(i\varepsilon)\right]+t-i\varepsilon\\
&=\exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\left(a_{[k]}-a_{[1]}-t+i\varepsilon\right)\right]\exp\left[\frac{1}{n}\left(\ln|\varepsilon|+\frac{\pi}{2}i\right)\right]+t-i\varepsilon\\
&\to\exp\left[\frac{1}{n}\sum_{k\ne\ell+1}^{n}\ln\left(a_{[k]}-a_{[1]}-t\right)\right]\lim_{\varepsilon\to0^+}\exp\left[\frac{1}{n}\left(\ln|\varepsilon|+\frac{\pi}{2}i\right)\right]+t\\
&=t
\end{aligned}
$$

as $\varepsilon\to0^+$. Hence, when $t=a_{[\ell+1]}-a_{[1]}$ for $1\le\ell\le n-1$, we have

$$
\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)=0.
$$

The proof of Lemma 2.3 is completed. □

3. The geometric mean is a Bernstein function

We now turn our attention to establishing an integral representation of the geometric mean $G_n(a+z)$ and to showing that the geometric mean is a Bernstein function.

Theorem 3.1. *Let $a=(a_1,a_2,\dots,a_n)$ with $a_k>0$ for $1\le k\le n$ and let $[a]$ denote the rearrangement of the sequence a in an ascending order, that is, $[a]=\left(a_{[1]},a_{[2]},\dots,a_{[n]}\right)$ and $a_{[1]}\le a_{[2]}\le\dots\le a_{[n]}$. For $z\in\mathbb{C}\setminus(-\infty,-\min\{a_k,1\le k\le n\}]$, the principal branch of the geometric mean $G_n(a+z)$ has the integral representation*

$$
G_n(a+z)=A_n(a)+z-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{[\ell]}}^{a_{[\ell+1]}}\left|\prod_{k=1}^{n}(a_k-t)\right|^{1/n}\frac{\mathrm{d}t}{t+z}, \tag{3.1}
$$

where $a+z=(a_1+z, a_2+z, \dots, a_n+z)$. *Consequently, the geometric mean* $G_n(a+t)$ *is a Bernstein function on* $(-\min\{a_k, 1\le k\le n\}, \infty)$.

Proof. By standard arguments, it is not difficult to see that

$$\lim_{z\to 0^+}[zh_n(z)]=0 \quad \text{and} \quad h_n(\overline{z})=\overline{h_n(z)}, \tag{3.2}$$

where $h_n(z)$ is defined by (2.4).

For any fixed point $z\in\mathbb{C}\setminus(-\infty,0]$, choose $0<\varepsilon<1$ and $r>0$ such that $0<\varepsilon<|z|<r$, and consider the positively oriented contour $C(\varepsilon,r)$ in $\mathbb{C}\setminus(-\infty,0]$ consisting of the half circle $z=\varepsilon e^{i\theta}$ for $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the half lines $z=x\pm i\varepsilon$ for $x\le 0$ until they cut the circle $|z|=r$, which close the contour at the points $-r(\varepsilon)\pm i\varepsilon$, where $0<r(\varepsilon)\to r$ as $\varepsilon\to 0$. See Figure 1.

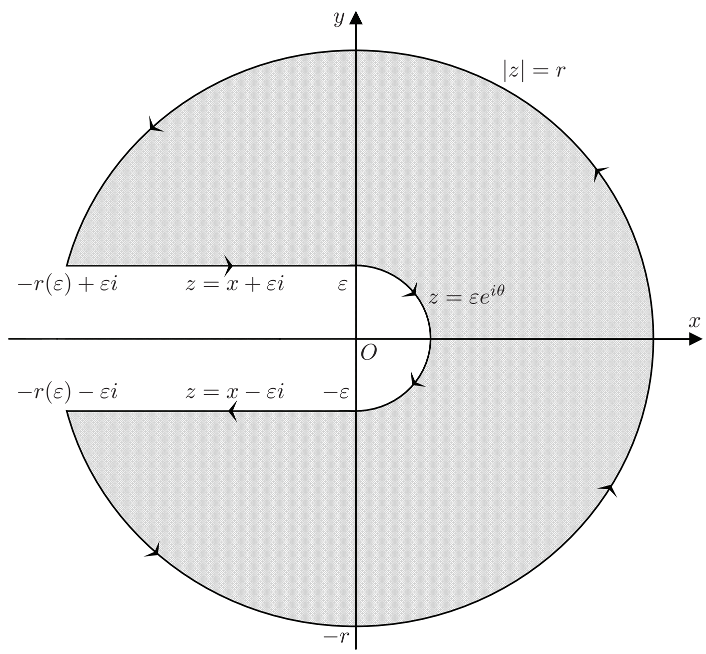

FIGURE 1. The contour $C(\varepsilon,r)$

By Cauchy integral formula, that is, Lemma 2.1, we have

$$\begin{aligned} h_n(z)&=\frac{1}{2\pi i}\oint_{C(\varepsilon,r)}\frac{h_n(w)}{w-z}\,\mathrm{d}w\\ &=\frac{1}{2\pi i}\left[\int_{\pi/2}^{-\pi/2}\frac{i\varepsilon e^{i\theta}h\big(\varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}-z}\,\mathrm{d}\theta+\int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]}\frac{ire^{i\theta}h\big(re^{i\theta}\big)}{re^{i\theta}-z}\,\mathrm{d}\theta\right.\\ &\quad\left.+\int_{-r(\varepsilon)}^{0}\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z}\,\mathrm{d}x+\int_{0}^{-r(\varepsilon)}\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z}\,\mathrm{d}x\right]. \end{aligned}\tag{3.3}$$

By the limit in (3.2), it follows that

$$\lim_{\varepsilon\to 0^+}\int_{\pi/2}^{-\pi/2}\frac{i\varepsilon e^{i\theta}h_n\big(\varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}-z}\,\mathrm{d}\theta=0. \tag{3.4}$$

By virtue of the limit (2.3) in Lemma 2.2, we deduce that

$$\begin{aligned}\lim_{\substack{\varepsilon\to0^+\\ r\to\infty}}\int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]}\frac{ire^{i\theta}h_n\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z}\,\mathrm{d}\theta&=\lim_{r\to\infty}\int_{-\pi}^{\pi}\frac{ire^{i\theta}h_n\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z}\,\mathrm{d}\theta\\&=2A_n\bigl([a]-a_{[1]}\bigr)\pi i,\end{aligned}\tag{3.5}$$

where $[a]-a_{[1]}=\bigl(0,a_{[2]}-a_{[1]},\dots,a_{[n]}-a_{[1]}\bigr)$. Utilizing the second formula in (3.2) and the limit (2.5) in Lemma 2.3 results in

$$\begin{aligned}&\int_{-r(\varepsilon)}^{0}\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z}\,\mathrm{d}x+\int_{0}^{-r(\varepsilon)}\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z}\,\mathrm{d}x\\
&=\int_{-r(\varepsilon)}^{0}\biggl[\frac{h_n(x+i\varepsilon)}{x+i\varepsilon-z}-\frac{h_n(x-i\varepsilon)}{x-i\varepsilon-z}\biggr]\,\mathrm{d}x\\
&=\int_{-r(\varepsilon)}^{0}\frac{(x-i\varepsilon-z)h_n(x+i\varepsilon)-(x+i\varepsilon-z)h_n(x-i\varepsilon)}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x\\
&=\int_{-r(\varepsilon)}^{0}\frac{(x-z)[h_n(x+i\varepsilon)-h_n(x-i\varepsilon)]-i\varepsilon[h_n(x-i\varepsilon)+h_n(x+i\varepsilon)]}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x\\
&=2i\int_{-r(\varepsilon)}^{0}\frac{(x-z)\Im h_n(x+i\varepsilon)-\varepsilon\Re h_n(x+i\varepsilon)}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x\\
&\to2i\int_{-r}^{0}\frac{\lim_{\varepsilon\to0^+}\Im h_n(x+i\varepsilon)}{x-z}\,\mathrm{d}x\\
&=-2i\int_{0}^{r}\frac{\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)}{t+z}\,\mathrm{d}t\\
&\to-2i\int_{0}^{\infty}\frac{\lim_{\varepsilon\to0^+}\Im h_n(-t+i\varepsilon)}{t+z}\,\mathrm{d}t\\
&=-2i\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{[\ell]}-a_{[1]}}^{a_{[\ell+1]}-a_{[1]}}\Biggl[\prod_{k=1}^{n}\bigl|a_{[k]}-a_{[1]}-t\bigr|\Biggr]^{1/n}\frac{\mathrm{d}t}{t+z}\end{aligned}\tag{3.6}$$

as $\varepsilon\to0^+$ and $r\to\infty$. Substituting equations (3.4), (3.5), and (3.6) into (3.3) and simplifying generate

$$h_n(z)=A_n\bigl([a]-a_{[1]}\bigr)-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{[\ell]}-a_{[1]}}^{a_{[\ell+1]}-a_{[1]}}\Biggl[\prod_{k=1}^{n}\bigl|a_{[k]}-a_{[1]}-t\bigr|\Biggr]^{1/n}\frac{\mathrm{d}t}{t+z}.\tag{3.7}$$

From (2.2) and (2.4), it is easy to obtain that

$$f_{a,n}(z)=h_n\bigl(z+a_{[1]}\bigr)+a_{[1]}.$$

Combining this with (3.7) and changing the variables of integrals, it is immediate to deduce that

$$\begin{aligned}f_{a,n}(z)&=A_n\bigl([a]-a_{[1]}\bigr)+a_{[1]}\\
&\quad-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{[\ell]}-a_{[1]}}^{a_{[\ell+1]}-a_{[1]}}\Biggl[\prod_{k=1}^{n}\bigl|a_{[k]}-a_{[1]}-t\bigr|\Biggr]^{1/n}\frac{\mathrm{d}t}{t+z+a_{[1]}}\\
&=A_n([a])-\frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\frac{\ell\pi}{n}\int_{a_{[\ell]}}^{a_{[\ell+1]}}\Biggl[\prod_{k=1}^{n}\bigl|a_{[k]}-t\bigr|\Biggr]^{1/n}\frac{\mathrm{d}t}{t+z},\end{aligned}$$

from which and the facts that

$$A_n([a]) = A_n(a) \quad \text{and} \quad \prod_{k=1}^{n} |a_{[k]} - t| = \prod_{k=1}^{n} |a_k - t|,$$

the integral representation (3.1) follows.

Differentiating with respect to z on both sides of (3.1) yields

$$G_n'(a+z) = 1 + \frac{1}{\pi} \sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{[\ell]}}^{a_{[\ell+1]}} \left[\prod_{k=1}^{n} |a_k - t|\right]^{1/n} \frac{\mathrm{d}t}{(t+z)^2},$$

which implies that $G_n'(a+t)$ is completely monotonic, and so the geometric mean $G_n(a+t)$ is a Bernstein function. Theorem 3.1 is proved. □

4. A new proof of the AG inequality

As an application of the integral representation (3.1) in Theorem 3.1, we can easily deduce the AG inequality (1.20) as follows.

Taking $z = 0$ in the integral representation (3.1) yields

$$G_n(a) = A_n(a) - \frac{1}{\pi} \sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{[\ell]}}^{a_{[\ell+1]}} \left[\prod_{k=1}^{n} |a_k - t|\right]^{1/n} \frac{\mathrm{d}t}{t} \le A_n(a), \tag{4.1}$$

from which the inequality (1.20) follows.

From (4.1), it is also immediate that the equality in (1.20) is valid if and only if $a_{[1]} = a_{[2]} = \cdots = a_{[n]}$, that is, $a_1 = a_2 = \cdots = a_n$. The proof of the AG inequality (1.20) is complete.

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